J. Fluid Mech. (1975), vol. 70, part 4, pp. 651–661
Printed in Great Britain


# Incompressible potential flow past 'not-so-slender' bodies of revolution at an angle of attack

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(Received 6 June 1974)

Using the method of matched asymptotic expansions, an expansion of the velocity potential for steady incompressible flow has been obtained to order $\epsilon^4$ for slender bodies of revolution at an angle of attack by representing the potential due to the body as a superposition of potentials of sources and doublets distributed along a segment of the axis inside the body excluding an interval near each end of the body. Also, expansions of the coefficients of longitudinal virtual mass and lateral virtual mass have been found. The pressure distributions over an ellipsoid of revolution of thickness ratio $\epsilon = 0{\cdot}3$ at zero angle of attack and at an angle of attack of 3° obtained by the present method are compared with results obtained from the exact theory and that of Van Dyke. The virtual-mass coefficients are also compared with those obtained from the exact theory and are found to be in good agreement up to $\epsilon = 0{\cdot}3$.

## 1. Introduction

The method of matched asymptotic expansions has already been applied to 'not-so-slender' bodies of revolution at angles of attack in incompressible potential flow. Lighthill (1948), Van Dyke (1951) and Broderick (1949) studied the supersonic problem by regular perturbation methods. Van Dyke (1959) has also given a second-order theory for incompressible and subsonic axisymmetric flows. Ashley & Landahl (1965) obtained the expansion for the potential for subsonic and supersonic axisymmetric flows to order $\epsilon^2$ and Cole (1968) found the expansion to order $\epsilon^4$ for the incompressible case using matched asymptotic expansions. The object of the present paper is to develop an asymptotic expansion for the solution of the Laplace equation for 'not-so-slender' bodies of revolution at an angle of attack by representing the potential due to the body as a superposition of potentials of sources and doublets distributed along a segment of the axis inside the body excluding an interval near each end of the body. To determine these intervals in which the source strength vanishes the procedure given by Handelsman & Keller (1967) is used. To expand the integrals for the potential the method given by Wang (1967) is followed. The expansion for the velocity potential is obtained to order $\epsilon^4$. The expansions for the virtual-mass coefficients are obtained, following Munk (1934), to order $\epsilon^2$. The pressure

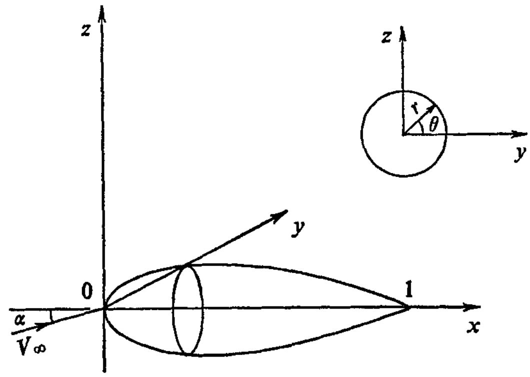


FIGURE 1. Co-ordinate system.

distributions over an ellipsoid of thickness ratio $\epsilon = 0{\cdot}3$ at zero angle of attack and at an angle of attack of 3° obtained by the present method are compared with values computed from the exact expression given by Matthews (1952) and values computed using Van Dyke's second-order theory without the addition of eigensolutions. The virtual-mass coefficients are compared with those given in Thwaites (1960).

## 2. Asymptotic expansion for $\Phi$

We shall consider incompressible flow around a slender body of revolution defined by $r = \epsilon R(x)$ for small values of the thickness ratio $\epsilon$; the uniform free stream is inclined from below at a small angle of attack $\alpha$ (figure 1). The full problem for the velocity potential is

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{rr} + r^{-1}\Phi_r + r^{-2}\Phi_{\theta\theta} = 0, \tag{1}$$

$$\Phi_r/\Phi_x = \epsilon R'(x) \quad \text{for} \quad r = \epsilon R(x) \quad \text{(tangency condition)}, \tag{2}$$

$$\Phi = V_\infty \cos(\alpha)\, x + V_\infty \sin(\alpha)\, r \sin\theta \quad \text{(upstream condition)}. \tag{3}$$

*The outer problem.* The system of equations becomes

$$\nabla^2\Phi^0 = 0, \tag{4}$$

$$\Phi^0 = Ux + Wr\sin\theta \quad \text{(upstream condition)}, \tag{5}$$

where $\Phi^0$ is the outer solution, $U = V_\infty \cos\alpha$ and $W = V_\infty \sin\alpha$. We shall consider an asymptotic expansion for the outer problem of the form

$$\Phi^0 = \phi_{00} + \epsilon^2\phi_{20} + \epsilon^4 \log(\epsilon)\, \phi_{41} + \epsilon^4\phi_{40} + \dots. \tag{6}$$

*The inner problem.* We introduce inner variables of the form

$$x = x, \quad \bar{r} = r/\epsilon. \tag{7}$$

The system of equations becomes

$$\Phi^i_{\bar{r}\bar{r}} + \bar{r}^{-1}\Phi^i_{\bar{r}} + \bar{r}^{-2}\Phi^i_{\theta\theta} + \epsilon^2\Phi^i_{xx} = 0, \tag{8}$$

$$\Phi^i_{\bar{r}} = \epsilon^2 R'(x)\, \Phi^i_x \quad \text{at} \quad \bar{r} = R(x) \quad \text{(tangency condition)}. \tag{9}$$

We introduce an inner expansion of the form

$$\Phi^i = \bar{\phi}_{00} + \epsilon\bar{\phi}_{10} + \epsilon^2 \log(\epsilon)\,\bar{\phi}_{21} + \epsilon^2\bar{\phi}_{20} + \epsilon^3 \log(\epsilon)\,\bar{\phi}_{31} + \epsilon^3\bar{\phi}_{30} + \epsilon^4 \log^2(\epsilon)\,\bar{\phi}_{42} + \epsilon^4 \log(\epsilon)\,\bar{\phi}_{41} + \epsilon^4\bar{\phi}_{40} + \ldots. \tag{10}$$

*First outer approximation* $\phi_{00}$. Substituting (6) in (4) and (5), we see that

$$\phi_{00} = Ux + Wr\sin\theta. \tag{11}$$

Introducing the inner variables and expanding for small $\epsilon$, we get

$$\phi_{00} = Ux + \epsilon W\bar{r}\sin\theta.$$

Therefore

1-term inner expansion of 1-term outer expansion $= Ux$.

*First inner approximation* $\bar{\phi}_{00}$. Substituting (10) in (8) and (9), we get

$$\bar{\phi}_{00\bar{r}\bar{r}} + \bar{r}^{-1}\bar{\phi}_{00\bar{r}} + \bar{r}^{-2}\bar{\phi}_{00\theta\theta} = 0; \quad \bar{\phi}_{00\bar{r}} = 0 \quad \text{at} \quad \bar{r} = R(x).$$

Matching with the outer solution requires

$$\bar{\phi}_{00} = Ux. \tag{12}$$

*Second outer approximation* $\phi_{20}$. Substitution of (6) in (4) and (5) gives

$$\nabla^2\phi_{20} = 0; \quad \phi_{20} = 0 \quad \text{upstream}.$$

We take the solution for $\phi_{20}$ to be of the form

$$\phi_{20} = -\frac{1}{4\pi}\int_a^b \frac{f_{20}(\xi)}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}\,d\xi + \frac{1}{4\pi}r\sin\theta\int_a^b \frac{g_{20}(\xi)}{[(x-\xi)^2+r^2]^{\frac{3}{2}}}\,d\xi, \tag{13}$$

where $f_{20}$ is the source intensity and $g_{20}$ the doublet intensity, the definitions of $a$ and $b$ being given in equations (A 2) in the appendix. Introducing the inner variables and expanding for small $\epsilon$ as in the appendix, we get

$$\begin{aligned}\phi_{20} = -\frac{1}{4\pi}\bigg\{&-2\log(\epsilon)f_{20} - 2\log(\bar{r})f_{20} + l_{20} + \epsilon^2\log(\epsilon)\frac{\bar{r}^2}{2}f''_{20}\\ &-\frac{\epsilon^2}{4}f_{20}\left[\frac{C_1}{x}+\frac{d_1}{1-x}\right] - \frac{\epsilon^2}{4}[C_1F_\xi(x,0) - d_1F_\xi(x,1)] + \epsilon^2\frac{\bar{r}^2}{2}\log(\bar{r})f''_{20}\\ &-\epsilon^2\frac{\bar{r}^2}{4}\frac{d^2}{dx^2}(2f_{20}+l_{20}) + \ldots\bigg\}\\ &+\frac{1}{4\pi}\sin\theta\left\{\frac{2g_{20}}{\epsilon\bar{r}} - \epsilon\log(\epsilon)\,g''_{20}\bar{r} - \epsilon g''_{20}\bar{r}\log\bar{r}\right.\\ &\left.+\epsilon\frac{\bar{r}d^2}{2dx^2}(g_{20}+k_{20}) + \ldots\right\}.\end{aligned} \tag{14}$$

Therefore

2-term inner expansion of 2-term outer expansion

$= Ux + \epsilon W\bar{r}\sin\theta + \epsilon g_{20}\sin(\theta)/2\pi\bar{r}$.

*Second inner approximation* $\bar{\phi}_{10}$

$$\bar{\phi}_{10\bar{r}\bar{r}} + \bar{r}^{-1}\bar{\phi}_{10\bar{r}} + \bar{r}^{-2}\bar{\phi}_{10\theta\theta} = 0; \quad \bar{\phi}_{10\bar{r}} = 0 \quad \text{at} \quad \bar{r} = R(x).$$

We take $\bar{\phi}_{10} = [A_{10}\bar{r} + B_{10}/\bar{r}]\sin\theta$. The boundary condition implies that

$$B_{10} = A_{10}R^2(x).$$

The 2-term inner expansion written in outer variables is

$$Ux + A_{10} r \sin\theta + \epsilon^2 A_{10} R^2(x) \sin(\theta)/r.$$

Expanding for small $\epsilon$, we obtain for the 2-term outer expansion of the 2-term inner expansion

$$Ux + A_{10} r \sin\theta + \epsilon^2 A_{10} R^2(x) \sin(\theta)/r.$$

Rewritten in inner variables, this becomes

$$Ux + \epsilon[A_{10}\bar{r} + A_{10}R^2(x)/\bar{r}]\sin\theta.$$

Now the matching condition

2-term outer expansion of 2-term inner expansion = 2-term inner expansion of 2-term outer expansion

gives $$Ux + \epsilon[A_{10}\bar{r} + A_{10}R^2(x)/\bar{r}]\sin\theta = Ux + \epsilon[W\bar{r} + g_{20}/2\pi\bar{r}]\sin\theta,$$

so that $$A_{10} = W, \quad g_{20}(x) = 2\pi A_{10}R^2(x) = 2\pi W R^2(x). \tag{15}$$

*Third inner approximation* $\bar{\phi}_{21}$

$$\bar{\phi}_{21\bar{r}\bar{r}} + \bar{r}^{-1}\bar{\phi}_{21\bar{r}} + \bar{r}^{-2}\bar{\phi}_{21\theta\theta} = 0; \quad \bar{\phi}_{21\bar{r}} = 0 \quad \text{at} \quad \bar{r} = R(x).$$

The matching condition

2-term outer expansion of 3-term inner expansion = 3-term inner expansion of 2-term outer expansion

gives $$\bar{\phi}_{21} = A_{21}(x) = f_{20}(x)/2\pi. \tag{16}$$

*Fourth inner approximation* $\phi_{20}$

$$\bar{\phi}_{20\bar{r}\bar{r}} + \bar{r}^{-1}\bar{\phi}_{20\bar{r}} + \bar{r}^{-2}\bar{\phi}_{20\theta\theta} = -\bar{\phi}_{00xx} = 0,$$

$$\bar{\phi}_{20\bar{r}} = R'(x)\,\bar{\phi}_{00x} = UR'(x) \quad \text{at} \quad \bar{r} = R(x).$$

Therefore $$\bar{\phi}_{20} = URR'\log\bar{r} + B_{20}(x). \tag{17}$$

The matching condition

2-term outer expansion of 4-term inner expansion = 4-term inner expansion of 2-term outer expansion

gives $$f_{20}/2\pi = URR',$$

$$B_{20} = -\frac{l_{20}}{4\pi} = -\frac{1}{4\pi}\left[f_{20}\log 4x(1-x) - \int_0^1 \frac{f_{20}(x) - f_{20}(\xi)}{|x-\xi|}\,d\xi\right]. \tag{18}$$

*Third outer approximation* $\phi_{41}$

$$\nabla^2\phi_{41} = 0; \quad \phi_{41} = 0 \quad \text{upstream}.$$

Therefore we take

$$\phi_{41} = -\frac{1}{4\pi}\int_a^b \frac{f_{41}(\xi)}{[(x-\xi)^2 + r^2]^{\frac{1}{2}}}\,d\xi + \frac{1}{4\pi} r\sin\theta \int_a^b \frac{g_{41}(\xi)}{[(x-\xi)^2 + r^2]^{\frac{3}{2}}}\,d\xi. \tag{19}$$

Rewriting this in inner variables and expanding for small $\epsilon$, we obtain

$$\phi_{41} = -\frac{1}{4\pi}\{-2\log(\epsilon)f_{41} - 2\log(\bar{r})f_{41} + l_{41} + \ldots\} + \frac{1}{4\pi}\sin\theta\left\{\frac{2g_{41}}{\epsilon\bar{r}} + \ldots\right\}. \quad (20)$$

*Fourth outer approximation* $\phi_{40}$

$$\nabla^2\phi_{40} = 0; \quad \phi_{40} = 0 \quad \text{upstream.}$$

Therefore we take

$$\phi_{40} = -\frac{1}{4\pi}\int_a^b \frac{f_{40}(\xi)}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}\,d\xi + \frac{1}{4\pi}r\sin\theta\int_a^b \frac{g_{40}(\xi)}{[(x-\xi)^2+r^2]^{\frac{3}{2}}}\,d\xi. \quad (21)$$

Rewriting in inner variables and expanding for small $\epsilon$, we get

$$\phi_{40} = -\frac{1}{4\pi}\{-2\log(\epsilon)f_{40} - 2\log(\bar{r})f_{40} + l_{40} + \ldots\} + \frac{1}{4\pi}\sin\theta\left\{\frac{2g_{40}}{\epsilon\bar{r}} + \ldots\right\}. \quad (22)$$

*Fifth inner approximation* $\bar{\phi}_{31}$

$$\phi_{31\bar{r}\bar{r}} + \bar{r}^{-1}\phi_{31\bar{r}} + \bar{r}^{-2}\bar{\phi}_{31\theta\theta} = 0; \quad \bar{\phi}_{31\bar{r}} = 0 \quad \text{at} \quad \bar{r} = R(x).$$

We take
$$\bar{\phi}_{31} = [A_{31}\bar{r} + B_{31}/\bar{r}]\sin\theta, \quad B_{31} = A_{31}R^2. \quad (23)$$

The matching condition

3-term outer expansion of 5-term inner expansion = 5-term inner expansion of 3-term outer expansion

gives
$$A_{31} = -g''_{20}/4\pi, \quad g_{41} = 2\pi B_{31} = -2\pi WR^2(RR')'.$$

*Sixth inner approximation* $\bar{\phi}_{30}$

$$\bar{\phi}_{30\bar{r}\bar{r}} + \frac{1}{\bar{r}}\bar{\phi}_{30\bar{r}} + \frac{1}{\bar{r}^2}\bar{\phi}_{30\theta\theta} = -\bar{\phi}_{10xx} = -\frac{g''_{20}}{2\pi}\frac{\sin\theta}{\bar{r}},$$

$$\bar{\phi}_{30\bar{r}} = R'\bar{\phi}_{10x} = R'\frac{g'_{20}}{2\pi}\frac{\sin\theta}{2\pi} \quad \text{at} \quad \bar{r} = R.$$

We write $\zeta = \bar{r}e^{i\theta}$. Therefore a particular solution for $\bar{\phi}_{30}$ is seen to be

$$\bar{\phi}_{30}^{(p)} = (-g''_{20}/16\pi i)[\zeta\log\bar{\zeta} - \bar{\zeta}\log\zeta].$$

Hence we take $\bar{\phi}_{30}$ to be of the form

$$\bar{\phi}_{30} = -\frac{g''_{20}}{16\pi i}[\zeta\log\bar{\zeta} - \bar{\zeta}\log\zeta] - \frac{g''_{20}}{16\pi i}[\zeta\log\bar{\zeta} - \bar{\zeta}\log\zeta] + A_{30}\bar{r}\sin\theta + B_{30}\frac{\sin\theta}{\bar{r}},$$

where the second term is added to make $\phi_{30}$ single valued and to match with the $\bar{r}\log\bar{r}$ term of the outer solution. $A_{30}$ and $B_{30}$ are determined by matching and the boundary condition

$$\bar{\phi}_{30} = -\frac{g''_{20}}{4\pi}\bar{r}\log(\bar{r})\sin\theta + A_{30}\bar{r}\sin\theta + B_{30}\frac{\sin\theta}{\bar{r}}. \quad (24)$$

Thus
$$A_{30} = (8\pi)^{-1}d^2[g_{20}+k_{20}]/dx^2$$

and
$$\frac{g_{40}}{2\pi} = B_{30} = R^2\left[A_{30} - \frac{g''_{20}}{4\pi}(1+\log R) - \frac{1}{2\pi}\frac{R'g'_{20}}{R}\right].$$

*Seventh inner approximation* $\bar{\phi}_{42}$

$$\bar{\phi}_{42\bar{r}\bar{r}}+\bar{r}^{-1}\bar{\phi}_{42\bar{r}}+\bar{r}^{-2}\bar{\phi}_{42\theta\theta}=0;\quad \bar{\phi}_{42\bar{r}}=0 \quad \text{at} \quad \bar{r}=R(x).$$

Matching with the outer solution requires

$$\bar{\phi}_{42}=f_{41}(x)/2\pi. \tag{25}$$

*Eighth inner approximation* $\bar{\phi}_{41}$

$$\bar{\phi}_{41\bar{r}\bar{r}}+\bar{r}^{-1}\bar{\phi}_{41\bar{r}}+\bar{r}^{-2}\bar{\phi}_{41\theta\theta}=-\bar{\phi}_{21xx}=-f''_{20}/2\pi,$$

$$\bar{\phi}_{41\bar{r}}=R'\bar{\phi}_{21x}=R'f'_{20}/2\pi \quad \text{at} \quad \bar{r}=R(x).$$

Therefore $$\bar{\phi}_{41}^{(p)}=-\frac{f''_{20}}{8\pi}\zeta\bar{\zeta}=-\frac{f''_{20}}{8\pi}\bar{r}^2.$$

We see that $$\bar{\phi}_{41}=(-f''_{20}/8\pi)\bar{r}^2+A_{41}\log\bar{r}+B_{41},$$

where $A_{41}=(R/4\pi)\{2R'f'_{20}+Rf''_{20}\}$ from the boundary condition, and

$$f_{41}=2\pi A_{41},\quad B_{41}=-(4\pi)^{-1}l_{41}+f_{40}/2\pi$$

from matching.

*Ninth inner approximation* $\bar{\phi}_{40}$

$$\bar{\phi}_{40\bar{r}\bar{r}}+\frac{1}{\bar{r}}\bar{\phi}_{40\bar{r}}+\frac{1}{\bar{r}^2}\bar{\phi}_{40\theta\theta}=-\bar{\phi}_{20xx}=-\frac{f''_{20}}{2\pi}\log\bar{r}-B''_{20},$$

$$\bar{\phi}_{40\bar{r}}=R'\bar{\phi}_{20x}=R'\frac{f'_{20}}{2\pi}\log R+B'_{20} \quad \text{at} \quad \bar{r}=R(x).$$

Thus $$\bar{\phi}_{40}^{(p)}=-\frac{f''_{20}}{16\pi}[\zeta\bar{\zeta}\log\zeta-\zeta\bar{\zeta}+\zeta\bar{\zeta}\log\bar{\zeta}-\zeta\bar{\zeta}]-\frac{1}{4\pi}B''_{20}\zeta\bar{\zeta}$$

$$=-\frac{f_{20}}{8\pi}\bar{r}^2\log\bar{r}+\frac{\bar{r}^2}{16\pi}\frac{d^2}{dx^2}(2f_{20}+l_{20}).$$

Therefore we take $$\bar{\phi}_{40}=\bar{\phi}_{40}^{p}+A_{40}\log\bar{r}+B_{40}.$$

The boundary condition gives

$$A_{40}=\frac{f'_{20}}{2\pi}RR'\log R+RR'B'_{20}+\frac{f''_{20}}{4\pi}R^2\log R-\frac{1}{8\pi}f''_{20}R^2+\tfrac{1}{2}B''_{20}R^2.$$

Matching gives $$f_{40}=2\pi A_{40},$$

$$B_{40}=-\frac{l_{40}}{4\pi}+\frac{f_{20}}{16\pi}\left[\frac{c_1}{x}+\frac{d_1}{1-x}\right]+\frac{1}{16\pi}[c_1F_\xi(x,0)-d_1F_\xi(x,1)].$$

Thus the inner expansion has been determined to order $\epsilon^4$. The pressure coefficient $C_p$ over the body is given by

$$C_p=(p-p_\infty)/\tfrac{1}{2}V_\infty^2=1-q^2/V_\infty^2,$$

where

$$\begin{aligned} q^2 &= (\Phi_x^i)^2+\left(\frac{1}{\epsilon}\Phi_{\bar{r}}^i\right)^2+\left(\frac{1}{\epsilon\bar{r}}\Phi_\theta^i\right)^2 \\ &= (\bar{\phi}_{00x}^2+\bar{r}^{-2}\bar{\phi}_{10\theta}^2)+2\epsilon\bar{\phi}_{00x}\bar{\phi}_{10x}+2\epsilon^2\log\epsilon[\bar{\phi}_{00x}\bar{\phi}_{21x}+\bar{r}^{-2}\bar{\phi}_{10\theta}\bar{\phi}_{31\theta}] \\ &\quad +\epsilon^2[2\bar{\phi}_{00x}\bar{\phi}_{20x}+\bar{\phi}_{10x}^2+\bar{\phi}_{20\bar{r}}^2+2\bar{r}^{-2}\bar{\phi}_{10\theta}\bar{\phi}_{30\theta}]+2\epsilon^3\log\epsilon[\bar{\phi}_{00x}\bar{\phi}_{31x}+\bar{\phi}_{10x}\bar{\phi}_{21x}] \\ &\quad +2\epsilon^3[\bar{\phi}_{00x}\bar{\phi}_{30x}+\bar{\phi}_{10x}\bar{\phi}_{20x}+\bar{\phi}_{20\bar{r}}\bar{\phi}_{30\bar{r}}]+\epsilon^4\log^2\epsilon[2\bar{\phi}_{00x}\bar{\phi}_{42x}+\bar{\phi}_{21x}^2+\bar{r}^{-2}\bar{\phi}_{31\theta}^2] \\ &\quad +2\epsilon^4\log\epsilon[\bar{\phi}_{00x}\bar{\phi}_{41x}+\bar{\phi}_{10x}\bar{\phi}_{31x}+\bar{\phi}_{21x}\bar{\phi}_{20x}+\bar{\phi}_{20\bar{r}}\bar{\phi}_{41\bar{r}}+\bar{r}^{-2}\bar{\phi}_{31\theta}\bar{\phi}_{30\theta}] \\ &\quad +\epsilon^4[2\bar{\phi}_{00x}\bar{\phi}_{40x}+2\bar{\phi}_{10x}\bar{\phi}_{30x}+\bar{\phi}_{20x}^2+2\bar{\phi}_{20\bar{r}}\bar{\phi}_{40\bar{r}}+\bar{\phi}_{30\bar{r}}^2+\bar{r}^{-2}\bar{\phi}_{30\theta}^2] \quad \text{with} \quad \bar{r}=R(x). \end{aligned}$$

## 3. Virtual-mass coefficients

To determine the virtual mass the method given by Munk (1934) is followed.

*Coefficient of longitudinal virtual mass*

The perturbation potential far from the body is given by

$$\phi = -\frac{1}{4\pi}\int_a^b \frac{F(\xi,\epsilon)\,d\xi}{[(x-\xi)^2+r^2]^{\frac{1}{2}}},$$

where $$F(\xi,\epsilon) = \epsilon^2 f_{20}(\xi)+\epsilon^4\log\epsilon f_{41}(\xi)+\epsilon^4 f_{40}(\xi)+\dots.$$

When $\phi$ is expanded for large $\omega \equiv [x^2+r^2]^{\frac{1}{2}}$, we get

$$\phi \approx -\frac{1}{4\pi\omega}\int_a^b F(\xi,\epsilon)\left[1+\frac{x\xi}{\omega^2}\right]d\xi$$
$$= -\frac{1}{4\pi}\frac{x}{\omega^3}\int_a^b \xi F(\xi,\epsilon)\,d\xi$$

since $$\int_a^b f(\xi,\epsilon)\,d\xi = \text{total source strength} = 0 \text{ for a closed body.}$$

This is the far-field potential for a doublet of strength

$$-\frac{1}{4\pi}\int_a^b \xi(\xi,\epsilon)\,d\xi = \mu_1(\epsilon).$$

Then the longitudinal virtual-mass coefficient $K_1$ is given by

$$\rho\tau K_1 = 4\pi\rho\mu_1/U-\rho\tau,$$

where $\tau$ is the volume of the body and $\rho$ its density. It can be shown that

$$\mu_1(\epsilon) = -\frac{1}{4\pi}\left\{\int_0^1 \xi F(\xi,\epsilon)\,d\xi-\tfrac{1}{4}\epsilon^4[d_1\bar{f}_{20}(1)+C_1\bar{f}_{20}(0)]\right\},$$

where $\bar{f}_{20}(\xi) = \xi f_{20}(\xi)$.

*Coefficient of lateral virtual mass*

Here the perturbation potential far from the body can be written as

$$\phi = \frac{1}{4\pi}\frac{z}{\omega^3}\int_a^b g(\xi,\epsilon)\,d\xi,$$

where $$g(\xi,\epsilon) = \epsilon^2 g_{20}(\xi)+\epsilon^4\log(\epsilon)\,g_{41}(\xi)+\epsilon^4 g_{40}(\xi)+\dots.$$

Hence the lateral virtual-mass coefficient $K_2$ is given by

$$\rho\tau K_2 = 4\pi\rho\mu_2/W-\rho\tau,$$

where $$\mu_2(\epsilon) = \frac{1}{4\pi}\int_a^b g(\xi,\epsilon)\,d\xi.$$

It can be shown that

$$\mu_2(\epsilon) = \frac{1}{4\pi}\int_0^1 g(\xi,\epsilon)\,d\xi.$$

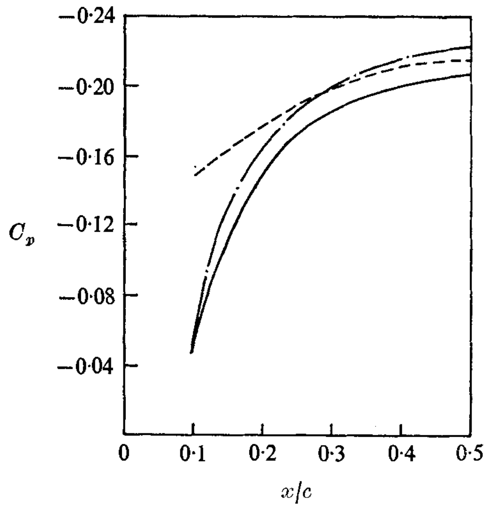


FIGURE 2. $C_p$ for an ellipsoid of revolution of thickness ratio $\epsilon = 0{\cdot}3$; $\alpha = 0$. ——, present results; —·—, exact results; – – – –, Van Dyke.

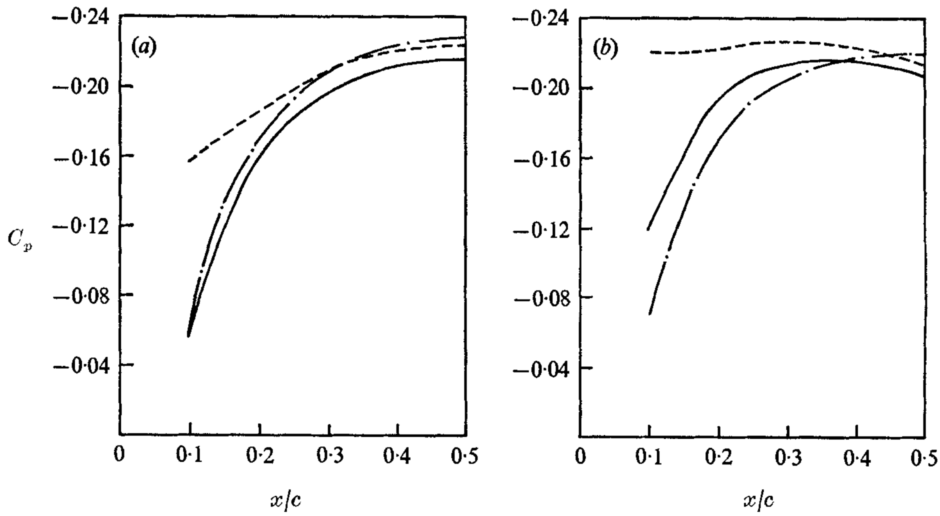


FIGURE 3. $C_p$ for an ellipsoid of revolution of thickness ratio $\epsilon = 0{\cdot}3$; $\alpha = 3°$. (a) $\theta = 0$. (b) $\theta = \frac{1}{2}\pi$. Curves as in figure 2.

## 4. Conclusion

It can be seen that the expansions for the velocity potential obtained by the present method for the paraboloid and ellipsoid coincide with the expansions obtained by Van Dyke (1959) when the eigensolutions are added. It is observed that the present technique provides a solution for the pressure distribution valid over a major portion of the surface for bodies of revolution with a thickness ratio of up to 30% in axisymmetric flows. The pressure distributions for the case of the ellipsoid have been calculated without incorporating any nose or

| | $K_1$ | | $K_2$ | |
|---|---|---|---|---|
| $\epsilon$ | Present results | Thwaites (1960) | Present results | Thwaites (1960) |
| 0 | 0·0000 | 0·0000 | 1·0000 | 1·0000 |
| 0·1 | 0·02 | 0·0207 | 0·9601 | 0·9602 |
| 0·2 | 0·0521 | 0·0591 | 0·8958 | 0·8943 |
| 0·3 | 0·0807 | 0·1054 | 0·8385 | 0·8259 |

TABLE 1. Coefficients of virtual mass for spheroids

tail corrections. The solution may be improved by using a local solution along the lines of Cole (1968). The calculations were stopped at $x/c = 0{\cdot}1$ since the expansions are not valid for $x/c = O(\epsilon^2)$. In figures 2 and 3 the pressure distributions over an ellipsoid of revolution whose meridian section is defined by

$$R(x) = \epsilon[x(1-x)]^{\frac{1}{2}}, \quad \epsilon = 0{\cdot}3,$$

are compared with exact results obtained from Matthews (1952) and with values calculated from Van Dyke (1951, 1959) without adding the eigensolutions. In table 1, virtual-mass coefficients for ellipsoids of different thickness ratios are compared with those obtained from the exact solution. It is seen that the expansion for the lateral virtual-mass coefficient is more accurate than that for the longitudinal virtual-mass coefficient.

The authors would like to thank the referees for important suggestions for improving the paper.

## Appendix. Expansion of $\phi$ for small $\epsilon$

$$\left.\begin{aligned} &\text{Let} \quad R^2(x) = S(x) = \sum_{n=1}^{\infty} C_n x^n, \quad \text{where} \quad C_n = \frac{S^{(n)}(0)}{n!}, \\ &\text{and} \quad S(x) = \sum_{n=1}^{\infty} d_n (1-x)^n, \quad \text{where} \quad d_n = (-1)^n \frac{S^{(n)}(1)}{n!}. \end{aligned}\right\} \tag{A 1}$$

Let $$a(\delta) = \sum_{n=1}^{\infty} a_n \delta^n, \quad b(\delta) = 1 - \sum_{n=1}^{\infty} b_n \delta^n, \quad \text{where} \quad \delta = \epsilon^2. \tag{A 2}$$

Consider $$\phi = -\frac{1}{4\pi}\int_{a(\delta)}^{b(\delta)} \frac{f(\xi)\, d\xi}{[(x-\xi)^2 + r^2]^{\frac{1}{2}}}. \tag{A 3}$$

Then we can define a function $\psi$ such that

$$\psi_x = -r\phi_r, \quad \psi_r = r\phi_x$$

and $$\psi = -\frac{1}{4\pi}\int_a^b \frac{(x-\xi)f(\xi)}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}\, d\xi. \qquad \text{(A 4)†}$$

† For a closed body the total source strength

$$\int_a^b F(\xi, \epsilon)\, d\xi = 0, \quad \text{where} \quad F(\xi, \epsilon) = \epsilon^2 f_{20}(\xi) + \epsilon^4 \log(\epsilon) f_{41}(\xi) + \epsilon^4 f_{40}(\xi) + \ldots.$$

This condition is used in deriving (A 4).

Now we can write

$$\psi = -\frac{1}{4\pi}\frac{\partial}{\partial x}\int_a^b f(\xi)\,[(x-\xi)^2+r^2]^{\frac{1}{2}}\,d\xi$$

$$= -\frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\int_a^b f(\xi)\left\{\frac{(x-\xi)\,[(x-\xi)^2+r^2]^{\frac{1}{2}}}{2}+\frac{r^2}{2}\log[(x-\xi)+[(x-\xi)^2+r^2]^{\frac{1}{2}}]\right\}d\xi.$$

Introducing the inner variable we write

$$\psi = -(4\pi)^{-1}\{I_1+I_2\},$$

where

$$I_1 = \frac{1}{2}\frac{\partial^2}{\partial x^2}\int_a^b f(\xi)\,(x-\xi)\,[(x-\xi)^2+\delta\bar{r}^2]^{\frac{1}{2}}\,d\xi,$$

$$I_2 = \frac{\delta\bar{r}^2}{2}\frac{\partial^2}{\partial x^2}\int_a^b f(\xi)\log\{(x-\xi)+[(x-\xi)^2+\delta\bar{r}^2]^{\frac{1}{2}}\}d\xi.$$

When $\xi = a$, in the integrand of $I_1$, $[(x-a)^2+\delta S(x)]^{\frac{1}{2}}$ becomes singular at $x = 0$, when expanded in terms of $\delta$. To make the expansion regular to the order considered we have to choose $a_1 = \frac{1}{4}C_1$. In a similar manner it can be shown that $b_1 = \frac{1}{4}d_1$.

Now

$$I_1 = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\lim_{t\to 0}\int_a^{x-t} f(\xi)\,(x-\xi)^2\left[1+\frac{1}{2}\frac{\epsilon^2\bar{r}^2}{(x-\xi)^2}+\dots\right]d\xi + \lim_{t\to 0}\int_b^{x+t} f(\xi)\,(\xi-x)^2\left[1+\frac{1}{2}\frac{\epsilon^2\bar{r}^2}{(\xi-x)^2}+\dots\right]d\xi\right\}$$

$$= \int_a^x f(\xi)\,d\xi+\int_b^x f(\xi)\,d\xi+\frac{\epsilon^2\bar{r}^2}{2}f'(x). \tag{A 5}$$

Proceeding in a similar manner it can be shown that

$$I_2 = \frac{\epsilon^2\bar{r}^2}{2}\left\{2\log 2-2\log\epsilon-2\log\bar{r}+\frac{d^2}{dx^2}\int_a^b f(\xi)\,\mathrm{sgn}\,(x-\xi)\log|x-\xi|\,d\xi\right\}. \tag{A 6}$$

The conditions $\psi_x = -r\phi_r$ and $\psi_r = r\phi_x$ imply that

$$\phi = -\frac{1}{4\pi}\left\{2\log\frac{2}{\epsilon\bar{r}}f(x)+\frac{d}{dx}\int_a^b f(\xi)\,\mathrm{sgn}\,(x-\xi)\log|x-\xi|\,d\xi - \frac{\epsilon^2\bar{r}^2}{2}f''(x)\log\frac{2}{\epsilon\bar{r}}-\frac{\epsilon^2\bar{r}^2}{2}f''(x)-\frac{\epsilon^2\bar{r}^2}{4}\frac{d^3}{dx^3}\int_a^b f(\xi)\,\mathrm{sgn}\,(x-\xi)\log|x-\xi|\,d\xi+\dots\right\}. \tag{A 7}$$

Now the potentials due to the doublets can be expanded as follows:

$$\frac{1}{4\pi}r\sin\theta\int_a^b\frac{g(\xi)\,d\xi}{[(x-\xi)^2+r^2]^{\frac{3}{2}}} = \frac{1}{4\pi}\frac{\sin\theta}{r}\frac{\partial}{\partial x}\int_a^b\frac{g(\xi)\,(x-\xi)\,d\xi}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}. \tag{A 8}$$

It can be seen that the integral in (A 8) is of the same form as the integral in (A 4).

It can be shown that

$$\frac{d}{dx}\int_a^b f(\xi)\,\mathrm{sgn}\,(x-\xi)\log|x-\xi|\,d\xi = f(x)\log[(x-a)\,(b-x)]-\int_a^b\frac{f(x)-f(\xi)}{|x-\xi|}\,d\xi.$$

We write $\log[(x-a)(b-x)]$ as $\log x - a/x + \ldots + \log(1-x) - (1-b)/(1-x) + \ldots$. Since $a = a_1\epsilon^2 = \frac{1}{4}c_1\epsilon^2$ and $b = 1 - b_1\epsilon^2 = 1 - \frac{1}{4}d_1\epsilon^2$ we get

$$\log(x-a)(b-x) = \log x(1-x) - \frac{\epsilon^2}{4}\left[\frac{c_1}{x} + \frac{d_1}{1-x}\right].$$

Now

$$\int_a^b \frac{f(x)-f(\xi)}{|x-\xi|}\,d\xi = \int_a^x \frac{f(x)-f(\xi)}{x-\xi}\,d\xi + \int_b^x \frac{f(x)-f(\xi)}{x-\xi}\,d\xi,$$

$$\begin{aligned}\int_a^x \frac{f(x)-f(\xi)}{x-\xi}\,d\xi &= \int_{a_1\epsilon^2}^x \frac{f(x)-f(\xi)}{x-\xi} = [F(x,\xi)]^x_{a_1\epsilon_2} = F(x,x) - F(x, a_1\epsilon^2)\\ &= F(x,x) - F(x,0) - a_1\epsilon^2 F_\xi(x,0) + \ldots\\ &= \int_0^x \frac{f(x)-f(\xi)}{x-\xi}\,d\xi - a_1\epsilon^2 F_\xi(x,0) + \ldots,\end{aligned}$$

where
$$F(x,\xi) = \int \frac{f(x)-f(\xi)}{x-\xi}\,d\xi.$$

Similarly
$$\int_b^x \frac{f(x)-f(\xi)}{x-\xi}\,d\xi = \int_1^x \frac{f(x)-f(\xi)}{x-\xi}\,d\xi + b_1\epsilon^2 F_\xi(x,1) + \ldots.$$

Therefore

$$\int_a^b \frac{f(x)-f(\xi)}{|x-\xi|}\,d\xi = \int_0^1 \frac{f(x)-f(\xi)}{|x-\xi|}\,d\xi + \tfrac{1}{4}\epsilon^2[d_1 F_\xi(x,1) - C_1 F_\xi(x,0)] + \ldots.$$

For convenience we write

$$l = f\log[4x(1-x)] - \int_0^1 \frac{f(x)-f(\xi)}{|x-\xi|}\,d\xi,$$

$$k = g\log[4x(1-x)] - \int_0^1 \frac{g(x)-g(\xi)}{|x-\xi|}\,d\xi.$$

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